

# Strength distributions and symmetry breaking in coupled microwave billiards

B. Dietz,<sup>1</sup> T. Guhr,<sup>2</sup> H.L. Harney,<sup>3</sup> and A. Richter<sup>1,\*</sup>

<sup>1</sup>*Institut für Kernphysik, Technische Universität Darmstadt, D-64289 Darmstadt, Germany*

<sup>2</sup>*Matematisk Fysik, LTH, Lunds Universitet, S-22100 Lund, Sweden*

<sup>3</sup>*Max-Planck-Institut für Kernphysik, D-69029 Heidelberg, Germany*

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## Abstract

Flat microwave cavities can be used to experimentally simulate quantum mechanical systems. By coupling two such cavities, we study the equivalent to the symmetry breaking in quantum mechanics. As the coupling is tunable, we can measure resonance strength distributions as a function of the symmetry breaking. We analyze the data employing a qualitative model based on Random Matrix Theory (RMT) and show that the results derived from the strength distribution are consistent with those previously obtained from spectral statistics.

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The breaking of quantum mechanical symmetries represents a prominent object of study in a rich variety of systems, ranging from high energy to condensed matter physics. Sometimes it is possible to determine the size of the symmetry breaking by analyzing statistical observables. We mention parity violation [1], the breaking of atomic and molecular symmetries [2, 3] and isospin mixing [4, 5, 6, 7, 8, 9] in nuclei. Symmetry breaking influences the spectral statistics as well as the wave function statistics, such as the distributions of partial widths and transition matrix elements. While symmetry breaking cannot be controlled or tuned in nuclei, atoms and molecules, this was possible for elastomechanical resonances in quartz crystals by successively breaking crystal symmetries [10]. An analogy to the nuclear case of symmetry breaking was given by two coupled microwave cavities [11] with a tunable coupling. The statistical properties of both systems are fully equivalent to those of a quantum system [12, 13]. Whereas these two investigations focused on the spectral statistics, width distributions for two different types of resonances in elastic aluminum plates were studied in Ref. [14] for a varying degree of mixing.

Here, we present experimental results on the distribution of products of partial widths in two coupled chaotic microwave cavities. First, we measure the distribution for different couplings. It can be normalized such that it only depends on the symmetry breaking. As this is different from Ref. [14], we can apply a qualitative statistical model which extends the model of Ref. [8] in order to, second, extract the size of the symmetry breaking from the data. We then show, third, that the symmetry breaking thereby obtained is consistent with the one previously found from the spectral statistics.

In the experiment, we used two flat cylindrical microwave cavities having the shape of the quarter of a Bunimovich stadium[11]. In both resonators (see Fig. 1) the radius of the quarter circle is 0.2 m. The ratios of the length of the rectangular part to the radius are  $\gamma_1 = 1$  and  $\gamma_2 = 1.8$ , respectively. Therefore the level densities increase with different slopes as functions of frequency [15, 16]. The cavities were put on top of each other and circular holes, 4 mm in diameter, were drilled through the walls of both resonators. The coupling was realized by a niobium pin, 2 mm in diameter, penetrating through the holes into both resonators. Due to the ring-shaped gap between the niobium pin and the hole surface, the pin acts like an antenna, which in the experimental frequency range supports one TEM-mode. The coupling is controlled by the penetration depth [11].

Information about the partial widths is obtained from transmission spectra. At a given

frequency  $f$ , the relative power transmitted from antenna  $a$  to antenna  $b$  is proportional to the absolute square of the scattering matrix element,  $P_{\text{out},b}/P_{\text{in},a} \sim |S_{ab}(f)|^2$ . For sufficiently isolated resonances, one has

$$S_{ab}(f) = \delta_{ab} - i \frac{\sqrt{\Gamma_{\mu a} \Gamma_{\mu b}}}{f - f_\mu + \frac{i}{2} \Gamma_\mu} \quad (1)$$

for  $f$  close to the frequency  $f_\mu$  of the  $\mu$ -th resonance. The quantities  $\Gamma_{\mu a}$  and  $\Gamma_{\mu b}$  are the partial widths related to the antennae  $a$  and  $b$ ,  $\Gamma_\mu$  is the total width of the resonance [17]. In the experiment, three antennae were attached to each resonator, where half of the microwave power was fed into each resonator (see insert of Fig. 2). Thereby, altogether six transmission spectra were obtained for 4 different couplings; in the order of increasing coupling these are denoted as (8,0), (5,3), (4,4) and (5,8) in Ref. [11]. Here, the pair  $(x_1, x_2)$  denotes the penetration depth in mm of the pin into either cavity. The transmission spectra were measured up to a maximum frequency of 17.5 GHz. The resonance strengths  $\Gamma_{\mu a} \Gamma_{\mu b}$ , i.e. the products of the partial widths, are determined as described in [18]. Resonances with peak heights below a certain value may not be detected, implying that some strength is missing in the tails of the distributions [18]. We will show that this has no effect on the results.

We work with the resonance strengths  $\Gamma_{\mu a} \Gamma_{\mu b}$  and their distributions. They yield the same information as the distributions of the partial widths  $\Gamma_{\mu a}$  themselves. The data are *unfolded* as in Refs. [17, 18].

To interpret the distribution of the empirical data  $\Gamma_{\mu a} \Gamma_{\mu b}$ , we employ a statistical model which is a special case of the Rosenzweig–Porter model [2]. A symmetry is associated with a quantum number. If it suffices to consider only two different values of it, the Hamiltonian can be written in the form

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} + \alpha \begin{pmatrix} 0 & V \\ V^T & 0 \end{pmatrix}, \quad (2)$$

where the first part preserves the symmetry and the second part breaks it with the parameter  $\alpha$ . For isospin mixing in nuclei,  $H_1$  and  $H_2$  correspond to the sub-Hamiltonians for two isospin quantum numbers, while  $V$  accounts for the Coulomb interaction that mixes isospin. In our case the “symmetry” preserving situation is simply realized by considering the states of each uncoupled cavity as eigenstates of a “symmetry operator”. Thus,  $H_1$  and  $H_2$  model the dynamics in the two cavities, without the coupling. We choose their dimensions  $N_1$  and

$N_2$  different because the level densities in the cavities have different slopes as a function of frequency. For later purposes, we introduce the *fractional densities*  $g_1 = N_1/(N_1 + N_2)$  and  $g_2 = N_2/(N_1 + N_2)$ . As the whole system is time-reversal invariant and the dynamics in the Bunimovich billiards is fully chaotic, we represent  $H_1$  and  $H_2$  by real-symmetric random matrices drawn from the Gaussian Orthogonal Ensemble (GOE) [12, 19, 20, 21]. The coupling is modeled by the off-diagonal blocks  $V$  and  $V^T$  in Eq. (2), where the matrix  $V$  is real with no symmetries and has dimension  $N_1 \times N_2$ . In the experiment the modes in one resonator are coupled to those in the other via one TEM-mode. We model the coupling of the  $N_1$  and  $N_2$  resonator modes with this TEM mode by an  $N_1$ -dimensional vector  $v$  and an  $N_2$ -dimensional vector  $w$ . The coupling matrix  $V$  then acquires the dyadic structure  $V = vw^T$ . The matrix  $V$  has rank  $M = 1$ . The entries of  $v$  and  $w$  are chosen as Gaussian random numbers. The distribution of the elements  $V_{nm}$  is

$$q(V_{nm}|\sigma) dV_{nm} = \frac{K_0(V_{nm}/\sqrt{\sigma})}{\pi\sqrt{\sigma}} dV_{nm}, \quad (3)$$

where  $K_0$  is the modified Bessel function of zeroth order [22]. The variance  $\sigma$  of the elements of  $V$  is adjusted exactly as described in [6, 8]. The vertical bar on the l.h.s. separates the argument of  $q$  into the random variable  $V_{nm}$  and the parameters, i.e. in this case  $\sigma$ . Universal spectral fluctuations have to be measured on the scale of the local mean level spacing  $D$ . Accordingly, the parameter measuring the size of the coupling or the symmetry breaking is  $\lambda = \alpha/D$  [23]. It connects to the spreading width via the relation  $\Gamma^\downarrow = 2\pi\alpha^2/D = 2\pi\lambda^2D$  [5]. *Spectral fluctuations* have been calculated perturbatively in  $\lambda$  in Refs. [24, 25] and exactly for the time-reversal non-invariant case in Refs. [6, 26]. For the analysis of the spectral properties of the coupled microwave billiards [11], a model Hamiltonian of the form Eq. (2) with a coupling matrix  $V$  of rank  $N_1$  was applied. While, in the relevant range of  $\lambda$ -values the spectral properties do not depend on the rank of  $V$ , the statistical properties of the eigenvector components deviate for  $M \ll N_1$ .

Accordingly we apply and extend the qualitative model of Ref. [8] where the statistics of *transition strengths* was considered. In a fully chaotic system, the partial width distribution converges to the Porter-Thomas form [12, 20, 21, 27]

$$\text{PT}(t_a|\tau_a) dt_a = \frac{1}{\sqrt{2\pi t_a/\tau_a}} \exp\left(-\frac{t_a}{2\tau_a}\right) \frac{dt_a}{\tau_a} \quad (4)$$

for a large level number. We write  $t_a$  for the partial width instead of  $\Gamma_{\mu a}$ , because the distribution does not depend on the resonance index  $\mu$ . In practice the distribution is

obtained from the sample of all resonances. Its first moment equals  $\tau_a$ . For two coupled chaotic systems the partial width distribution  $p(t_a|\lambda, \tau_a)$  involves the symmetry breaking Hamiltonian in Eq. (2) for large level numbers. It is easily constructed in the limiting case without symmetry breaking; i.e. for  $\lambda = 0$  we have

$$p(t_a|\lambda = 0, \tau_a) = g_1^2 \text{PT} \left( t_a \left| \frac{\tau_a}{g_1} \right. \right) + g_2^2 \text{PT} \left( t_a \left| \frac{\tau_a}{g_2} \right. \right) + 2g_1g_2\delta \left( \frac{t_a}{\tau_a} \right). \quad (5)$$

The distribution in Eq. (5) again has the expectation value  $\tau_a$ . We now model the case of small symmetry breaking  $\lambda$  by an interpolating ansatz: We expect the Porter-Thomas distributions in Eq. (5) to maintain their shape, such that only their width parameters change. The delta function in Eq. (5) acquires a width and develops into a non-singular function. Numerical studies have lead us to approximate it by [28]  $P_0(t|\rho) dt = K_0(\sqrt{t/\rho}) / (\pi\sqrt{t\rho}) dt$ , where  $\rho$  stands for the variance parameter. We arrive at the model

$$p(t_a|\lambda, \tau_a) = g_1^2 \text{PT} (t_a|\tau_a \kappa_1(\eta)) + g_2^2 \text{PT} (t_a|\tau_a \kappa_2(\eta)) + 2g_1g_2 P_0(t_a|\tau_a \eta^2). \quad (6)$$

The shape of the whole distribution is determined by the quantities  $\kappa_1(\eta)$ ,  $\kappa_2(\eta)$  and  $\eta = \eta(\lambda)$  which all depend on  $\lambda$ . With the help of the limiting case Eq. (5), we can come up with educated guesses for these functions. We must have  $\eta(0) = 0$  and furthermore  $\kappa_j(0) = 1/g_j$ ,  $j = 1, 2$ . As the functions  $\kappa_j$  ought to be even in  $\eta$ , we choose  $\kappa_j(\eta) = 1/g_j + (1 - 1/g_j)\eta^2$ ,  $j = 1, 2$ . To construct the function  $\eta = \eta(\lambda)$ , we fit the ansatz given in Eq. (6) including these choices to Monte Carlo simulations of the distributions involving the Hamiltonian in Eq. (2) for  $N_1 + N_2 = 300$ . We do that for 21 values of  $\lambda$  in the interval  $[0.05, 0.25]$  which is the relevant parameter range in the experiment. The fractional densities in the experiment are  $g_1 = 0.59$  and  $g_2 = 0.41$ . In Fig. 2 we display the values thus obtained for  $\eta(\lambda)$ . The functional form is well described by the polynomial  $\eta(\lambda) = 2.57\lambda - 1.98\lambda^2$ .

From the distributions for two partial widths  $t_a$  and  $t_b$ , say, we obtain the distribution for the resonance strength  $y = t_a t_b$  as in [18] by a filter integration. As we know the fractional densities  $g_1$  and  $g_2$  in the experiment, we arrive at a qualitative model for the resonance strength distribution that depends only on one single parameter — the measure  $\lambda$  for the symmetry breaking. The distributions in Eq. (6) and accordingly the strength distribution

diverge for  $t_a \rightarrow 0$ . Thus, for their graphical representation we use the logarithmic variable  $z = \log_{10}(y/\tau_a\tau_b)$ .

Despite its simplicity, the qualitative model in Eq. (6) yields a satisfactory description. We demonstrate this in Fig. 3 where a Monte Carlo simulation of resonance strength distributions obtained from the random matrix Hamiltonian in Eq. (2) with the same fractional densities as in the experiment is compared to the calculated strength distribution for four different values of  $\lambda$ . The strongest deviations between both curves are observed for the smallest value of  $\lambda$ . For values  $\lambda < 0.03$  the description by the qualitative model ceases to be satisfactory. But, most importantly, the position of the maximum is described very well in all four examples. It is particularly this feature which makes the qualitative model in Eq. (6) useful for the analysis of experimental data. The fit of the model to the experimental data is displayed in Fig. 4. Due to the above mentioned missing strength in the left tails, the shape of the distribution is not described quantitatively. In fact, for the determination of  $\lambda$  only strengths in a  $z$  interval [-3,1.5] were taken into account, where the probability of missing strength is small. The corresponding experimental distributions are shown together with the RMT model fits in the insets of Fig. 4 b)-d). For very small values of  $\lambda$  (cf. Fig. 4 a)) the distributions agree fairly well in this interval. We emphasize again that the peak positions in the chosen interval of  $z$  values uniquely determine the coupling strengths. We thus expect reliable results for the parameter  $\lambda$ . To carefully check how much the missing strength influences the extracted values of  $\lambda$ , we amended the qualitative model by taking care of the experimental thresholds using an analytic ansatz [18]. As expected, the influence of the thresholds on the extracted values of  $\lambda$  turned out to be negligible, except when  $\lambda$  is essentially zero. As only the position of the maximum in the strength distribution is relevant, we have some freedom in choosing the distribution  $P_0$  in Eq. (6). We also tried a Porter-Thomas distribution, but the  $K_0$ -one describes the shape of the resonance strength distribution better.

We finally compare the coupling strengths  $\lambda$  found in the present work with those extracted from the *spectral correlations* in Ref. [11, 29]. The values are given in Tab. I. The  $\lambda$  values obtained in the present work are averaged over all six antennae combinations. The two analyses agree within the experimental errors. For the weakest coupling only an upper limit can be given. This is due to the fact that for essentially zero coupling a considerable share of strength lies below the experimental threshold of detection. The consistency

for the other couplings is an encouraging corroboration of the analysis carried out in this contribution [30].

In conclusion, we have measured resonance strength distributions for two coupled microwave billiards modeling quantum systems with symmetry breaking. We analyzed the data with a qualitative model which depends only on one single free parameter — the size of the symmetry breaking. Our results are interesting from an additional point of view. The spectral correlations are more strongly affected by missing levels than partial widths and transition or resonance strength distributions are affected by missing strength. This is so because the latter do not comprise *correlations*, they are just *densities*. Thus the observables related to the wave functions may provide more reliable information than the spectral correlations. We have shown, with data much better than are usually available, that the empirical information extracted from the wave function observables is consistent with that obtained from the spectral correlations.

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\* Electronic address: richter@ikp.tu-darmstadt.de

- [1] J. D. Bowman *et al.*, Annu. Rev. Nucl. Part. Sci. **43**, 829 (1993); G.E. Mitchell, J.D. Bowman, and H.A. Weidenmüller, Rev. Mod. Phys. **71**, 445 (1999).
- [2] N. Rosenzweig and C. E. Porter, Phys. Rev. **120**, 1698 (1960).
- [3] E. Haller, H. Köppel, and L. S. Cederbaum, Chem. Phys. Lett. **101**, 215 (1983).
- [4] G. E. Mitchell *et al.*, Phys. Rev. Lett. **61**, 1473 (1988).
- [5] H. L. Harney, A. Richter, and H. A. Weidenmüller, Rev. Mod. Phys. **58**, 607 (1986).
- [6] T. Guhr and H. A. Weidenmüller, Ann. Phys. (N.Y.) **199**, 412 (1990).
- [7] J. F. Shriner, Jr., G. E. Mitchell, and B. A. Brown, Phys. Rev. C **71**, 024313 (2005), and refs. therein.

- [8] C.I. Barbosa, T. Guhr, and H.L. Harney, Phys. Rev. E **62**, 1936 (2000).
- [9] S. Åberg *et al.*, Phys. Lett. B **598**, 42 (2004).
- [10] C. Ellegaard *et al.*, Phys. Rev. Lett. **77**, 4918 (1996).
- [11] H. Alt *et al.*, Phys. Rev. Lett. **81**, 4847 (1998).
- [12] H.- J. Stöckmann, *Quantum Chaos - An Introduction*, (Cambridge University Press, Cambridge, 1999).
- [13] A. Richter, in: *Emerging Applications of Number Theory, The IMA Volumes in Mathematics and its Applications*, Vol. **109**, edited by D.A. Hejhal *et al.*, p. 479, (Springer, New York, 1999).
- [14] A. Andersen *et al.*, Phys. Rev. E **63**, 066204 (2001).
- [15] H. Weyl, *J. Reine Angew. Math.* **141**, 1 (1912); *ibid* 163, *J. Reine Angew. Math.* **143**, 177 (1913).
- [16] H. P. Baltes and E. R. Hilf, *Spectra of Finite Systems* (Bibliographisches Institut Mannheim, 1975).
- [17] H. Alt *et al.*, Phys. Rev. Lett. **74**, 62 (1995).
- [18] C. Dembowski *et al.*, Phys. Rev. E **71**, 046202 (2005).
- [19] M. L. Mehta, *Random Matrices* (Academic Press, New York, 1991), 2nd ed.
- [20] T. Guhr, A. Müller-Groeling, and H.A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
- [21] F. Haake, *Quantum Signatures of Chaos*, 2nd edition, (Springer Verlag, Berlin 2001).
- [22] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products.*, (Academic Press, New York, 1980).
- [23] F.J. Dyson, J. Math. phys. **3**, 1191 (1962); A. Pandey, Ann. Phys. (NY) **134**, 110 (1981).
- [24] D. M. Leitner, Phys. Rev. E **48**, 2536 (1993); D. M. Leitner, H. Köppel, and L. S. Cederbaum, Phys. Rev. Lett. **73**, 2970 (1994).
- [25] J.B. French *et al.*, Ann. Phys. (NY) **181**, 198 (1988).
- [26] A. Pandey, Chaos, Solitons and Fractals **5**, 1275 (1995).
- [27] C.E. Porter, *Statistical Theories of Spectra: Fluctuations*, (Academic Press, New York, 1965).
- [28] For very weak symmetry breaking, this follows from the  $K_0$ -distribution of the matrix elements  $V_{nm}$ .
- [29] A. Y. Abul-Magd *et al.*, Phys. Rev. E **65**, 056221 (2002).
- [30] However, this consistency does not prove that the dyadic coupling specified prior to Eq. (3) is

necessary to describe the present data. It is required by the experimental set-up. We have convinced ourselves that the qualitative model in the version of reference [8] with high-dimensional coupling yields this consistency, too.

- [31] C. Dembowski, Dissertation D17, TU-Darmstadt (2003).

## **Figures**

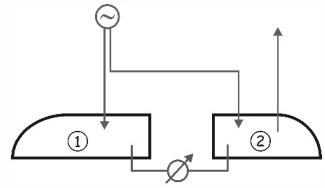


FIG. 1: The two cavities are coupled with a tunable coupling

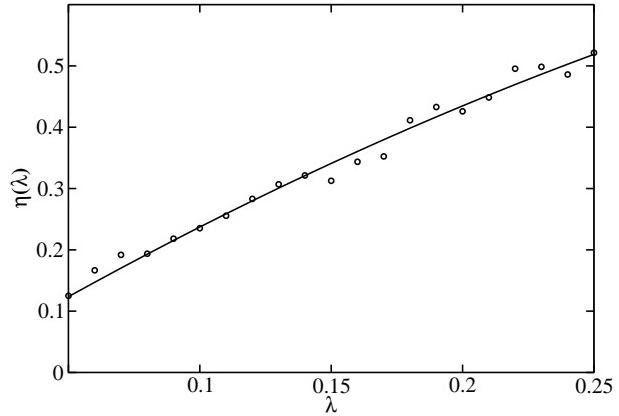


FIG. 2: Size of the tunable coupling  $\lambda$  between the two cavities sketched in Fig. 1. The function  $\eta(\lambda)$  enters Eq. (6).

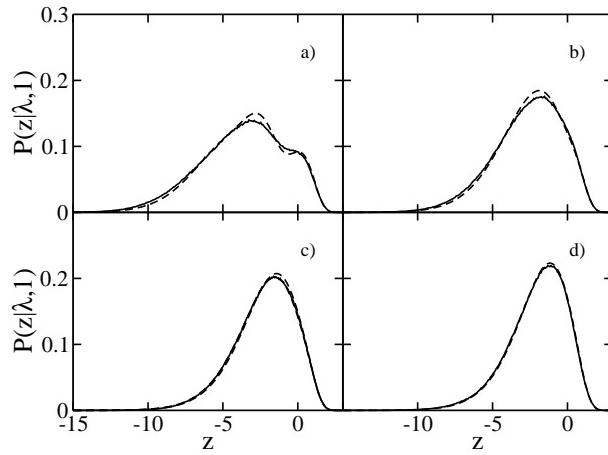


FIG. 3: Monte Carlo simulation of resonance strength distributions (solid lines) compared to the calculated strength distribution (dashed lines) for the couplings  $\lambda = 0.03$  (a),  $\lambda = 0.10$  (b),  $\lambda = 0.17$  (c),  $\lambda = 0.24$  (d).

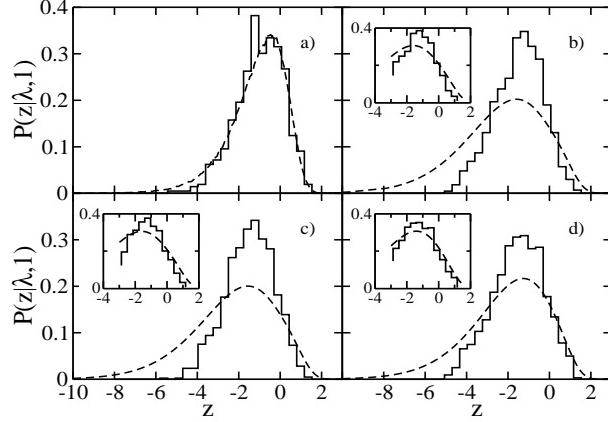


FIG. 4: The experimental resonance strength distributions (histograms) for the couplings (8,0) (a), (5,3) (b), (4,4) (c) and (5,8) (d) fitted to the calculated strength distribution. The symmetry breaking parameter  $\lambda$  equals  $\lambda = 0$  in (a),  $\lambda = 0.110$  in (b),  $\lambda = 0.125$  in (c), and  $\lambda = 0.185$  in (d). The insets in b), c), and d) show experimental strength distributions together with RMT model fits for a  $z$  interval  $[-3, 1.5]$  in which the probability of missing strength is small.

## Tables

TABLE I: Symmetry breaking parameter  $\lambda$  from the spectral correlations and from the resonance strength distributions.

| Physical coupling | $\lambda$<br>Ref. [11] | $\lambda$<br>Present work |
|-------------------|------------------------|---------------------------|
| (8,0)             | $\leq 0.029$           | $< 0.003$                 |
| (5,3)             | $0.105 \pm 0.008$      | $0.116 \pm 0.003$         |
| (4,4)             | $0.130 \pm 0.007$      | $0.122 \pm 0.004$         |
| (5,8)             | $0.180 \pm 0.006$      | $0.195 \pm 0.007$         |